Problem 1

$$I = \int_V d\bar{r}\, e^{-\mu r} \left(\bar{\nabla} \cdot \frac{\hat{r}}{r^2} \right), \qquad V\text{: sphere with radius } R$$

1. use Eq. (1.99) which is the Gauss law in differential form for a point charge

$$\bar{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi\, \delta^3(\bar{r})$$

$$\rightarrow \quad I = \int_V d\bar{r}\, e^{-\mu r}\, 4\pi\, \delta^3(\bar{r}) = \underline{4\pi}$$

2. use Eq. (1.59)

$$\int_V d\bar{r}\, f\,(\bar{\nabla} \cdot \bar{A}) = - \int_V d\bar{r}\, \bar{A} \cdot (\bar{\nabla} f) + \oint_S d\bar{a} \cdot f \bar{A}$$

$$\rightarrow \quad I = - \int_V d\bar{r} \left(\frac{\hat{r}}{r^2} \right) \cdot \left(\bar{\nabla} e^{-\mu r} \right) + \oint_S d\bar{a} \cdot \left[\frac{e^{-\mu r}\, \hat{r}}{r^2} \right]$$

$$= - \int_V d\bar{r} \left(\frac{\hat{r}}{r^2} \right) \cdot \left(-\mu\, e^{-\mu r}\, \hat{r} \right) + 4\pi\, e^{-\mu R}$$

$$= \int_V d\bar{r} \left(\mu \frac{e^{-\mu r}}{r^2} \right) + 4\pi e^{-\mu R} = \int_V r^2 dr\, d\Omega \; \mu \frac{e^{-\mu r}}{r^2} + 4\pi e^{-\mu R}$$

$$= \mu\, 4\pi \int_0^R dr \left(e^{-\mu r} \right) + 4\pi e^{-\mu R} = 4\pi \mu \frac{1}{\mu} \left\{ 1 - e^{-\mu R} \right\} + 4\pi e^{-\mu R}$$

$$= \underline{4\pi}$$

Problem 3

(a) $$\bar{F} = \underbrace{(\bar{r} \times \bar{p})}_{\text{vector}} \underbrace{(\bar{r} \cdot \bar{p})}_{\text{Scalar}}$$

use $$\bar{\nabla} \cdot (f \bar{A}) = f (\bar{\nabla} \cdot \bar{A}) + \bar{A} \cdot (\bar{\nabla} f)$$

$$\rightarrow \quad \bar{\nabla} \cdot \bar{F} = (\bar{r} \cdot \bar{p}) \left[\bar{\nabla} \cdot (\bar{r} \times \bar{p}) \right] + (\bar{r} \times \bar{p}) \cdot \left[\bar{\nabla} (\bar{r} \cdot \bar{p}) \right]$$

and

$$\bar{\nabla} (\bar{r} \cdot \bar{p}) = \bar{p} \times (\bar{\nabla} \times \bar{r}) + (\bar{r} \cdot \bar{\nabla}) \bar{p} + (\bar{p} \cdot \bar{\nabla}) \bar{r} = (\bar{p} \cdot \bar{\nabla}) \bar{r}$$

$$\bar\nabla\cdot(\bar r\times\bar p) = \bar p\cdot(\bar\nabla\times\bar r) - \bar r\cdot(\bar\nabla\times\bar p) = 0$$

$$(\bar p\cdot\bar\nabla)\bar r = \bar p$$

$$\rightarrow \quad \bar\nabla\cdot\bar F = (\bar r\times\bar p)\cdot\bar p = 0 \qquad \text{as} \quad \bar r\times\bar p \perp \bar p$$

$$\bar\nabla\times\bar F$$

use $\quad \bar\nabla\times(f\bar A) = f(\bar\nabla\times\bar A) - \bar A\times(\bar\nabla f)$

$$\rightarrow \quad \bar\nabla\times\bar F = (\bar r\cdot\bar p)\left[\bar\nabla\times(\bar r\times\bar p)\right] - (\bar r\times\bar p)\times\left[\bar\nabla(\bar r\cdot\bar p)\right]$$

again $\quad \bar\nabla(\bar r\cdot\bar p) = (\bar p\cdot\bar\nabla)\bar r = \bar p$

$$\bar\nabla\times(\bar r\times\bar p) = (\bar p\cdot\bar\nabla)\bar r - \underbrace{(\bar r\cdot\bar\nabla)\bar p}_{0} + \underbrace{\bar r(\bar\nabla\cdot\bar p)}_{0} - \bar p\underbrace{(\bar\nabla\cdot\bar r)}_{3}$$

$$= \bar p - 3\bar p = -2\bar p$$

$$\rightarrow \quad \bar\nabla\times\bar F = -(\bar r\cdot\bar p)2\bar p - (\bar r\times\bar p)\times\bar p$$

$$= -2\bar{p}(\bar{r}\cdot\bar{p}) + \bar{p}\times(\bar{r}\times\bar{p})$$

$$= -2\bar{p}(\bar{r}\cdot\bar{p}) + p^2\bar{r} - \bar{p}(\bar{p}\cdot\bar{r})$$

$$= -3\bar{p}(\bar{r}\cdot\bar{p}) + p^2\bar{r}$$

Problem 2

$$\phi(\bar{r}) = q\, e^{-\mu r}/r \,, \qquad r = |\bar{r}|$$

μ: constant

(a) $\bar{E} = -\bar{\nabla}\phi(\bar{r})$ 3D spherical coordinates

$$\rightarrow \bar{E} = -q\frac{\partial}{\partial r}\left[\frac{e^{-\mu r}}{r}\right]\hat{r}$$

$$= \hat{r}\, q\, e^{-\mu r}\left[\frac{\mu}{r} + \frac{1}{r^2}\right]$$

radial field

Notice that μ helps us to check the dimension we notice that E dies of quicker than for a single point charge as r approches infinity. This indicates that the total charge might be zero, but not all of it is in a single point.

6 find $\rho(\bar{r})$

From E we can conclude that it has a radial symmetry, spherical. we can "turn around" the differential form of the Gauss theorem

$$\rho(\bar{r}) = \epsilon_0 \bar{\nabla} \cdot \bar{E}$$

$$= \epsilon_0 q \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ e^{-\mu r} (\mu r + 1) \right\}$$

$$= \epsilon_0 q \frac{1}{r^2} \left[-\mu(\mu r + 1) + \mu \right] e^{-\mu r}$$

$$= \epsilon_0 \frac{q}{r^2} \left[-\mu^2 r \right] e^{-\mu r} = -\epsilon_0 \frac{q\mu^2}{r} e^{-\mu r}$$

..., but check the "total charge" according to this distribution $\rho(\bar{r})$

$$Q = \int_V d\bar{r}\,\rho(\bar{r}) = -\epsilon_0 q \mu^2 \int_0^\infty r^2 dr\, d\Omega\, \frac{e^{-\mu r}}{r}$$

$$= -4\pi\epsilon_0 q \mu^2 \int_0^\infty r\,dr\, e^{-\mu r} \qquad = -4\pi\epsilon_0 q \mu^2 \frac{1}{\mu^2}$$

$$= -4\pi\epsilon_0 q$$

which is negative if $q > 0$

(c)

This does not allow the total E to die off exponentially as $r \to \infty$

so we need to add a point charge at $r = 0$, ρ could not be determined at 0 by our method, without a refinement.

$$\rho_{Total} = \rho(\bar{r}) + \epsilon_0 q\, \delta^3(\bar{r}) = \rho(\bar{r}) + \epsilon_0 q \frac{\delta(r)}{r^2}$$

This has relevance to a hydrogen atom in the ground state, for example....