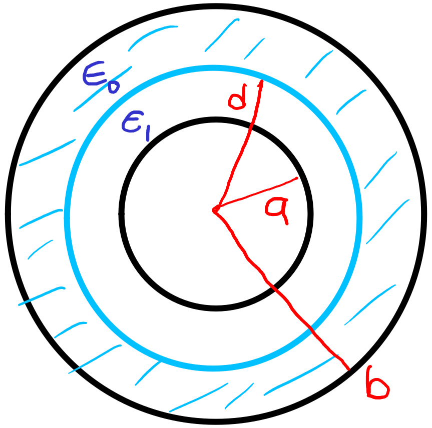
Problem 1



a) Find the capacity per length

Assume V_0 between plates. We need to connect V_0 and the charge on a plate, assume $Q_a = Q$, no electret, radial symmetry

$$\oint \bar{D} \cdot d\bar{s} = Q_a$$

$$\rightarrow 2\pi r L \bar{D} \cdot \hat{a}_r = Q = \lambda L \quad \rightarrow \quad \bar{D}(r) = \frac{\hat{a}_r \lambda}{2\pi r}, \qquad Q = CV_0$$

$\bar{D}(r)$ gildir fyrir $a < r < b$

$\lambda = C' V_0$

$C = C' L$

$$V_0 = V_b - V_a = -\int_a^b \bar{E} \cdot d\bar{l}$$

$$\bar{E}(r) = \begin{cases} \hat{a}_r \dfrac{\lambda}{2\pi \epsilon_1 r} & \underline{a < r < d} \\[2ex] \hat{a}_r \dfrac{\lambda}{2\pi \epsilon_0 r} & \underline{d < r < b} \end{cases}$$

$$Q = \lambda L$$

L: Length of cylinder

λ: charge per length

$L \rightarrow \infty$

2

$$V_0 = -\int_a^b \bar{E}\cdot d\bar{l} = -\int_a^d dr \frac{\lambda}{2\pi\epsilon_1 r} - \int_d^b dr \frac{\lambda}{2\pi\epsilon_0 r}$$

$$= -\frac{\lambda}{2\pi}\left\{ \int_a^d \frac{dr}{\epsilon_1 r} + \int_d^b \frac{dr}{\epsilon_0 r} \right\} = -\frac{\lambda}{2\pi}\left\{ \frac{1}{\epsilon_1}\ln\left(\frac{d}{a}\right) + \frac{1}{\epsilon_0}\ln\left(\frac{b}{d}\right) \right\}$$

$$= -\frac{\lambda}{2\pi}\left\{ \frac{1}{\epsilon_1}\ln\left(\frac{d}{a}\right) + \frac{1}{\epsilon_0}\ln\left(\frac{b}{d}\right) \right\}$$

$$\underline{C'} = \frac{\lambda}{|V_0|} = \frac{2\pi}{\frac{1}{\epsilon_1}\ln\left(\frac{d}{a}\right) + \frac{1}{\epsilon_0}\ln\left(\frac{b}{d}\right)}$$

capacitance per length →

$$= \boxed{\frac{2\pi\epsilon_0}{\frac{\epsilon_0}{\epsilon_1}\ln\left(\frac{d}{a}\right) + \ln\left(\frac{b}{d}\right)}}.$$

b) Find equivalent charges

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} \longrightarrow \bar{P} = \bar{D} - \epsilon_0 \bar{E}$$

and

$$\rho_{ps} = \bar{P} \cdot \hat{a}_n \,, \qquad \rho_p = -\bar{\nabla} \cdot \bar{P}$$

outer metal surface at $r = a$ and inner metal surface at $r = b$ carry charge of magnitude $|Q|$. These are free charges. In cylindrical coordinates

$$\bar{\nabla} \cdot P(r) = \frac{1}{r} \frac{\partial}{\partial r} \left(r\, P(r) \right)$$

for our E above this will give $\rho_p = 0$

so, we have no bulk equivalent charges between the cylindrical metal plates, but the dielectric cylinders will carry equivalent bound charges at their surfaces:

But the outer cylinder with ϵ_0 has none, as there $P(r) = 0$

$$\bar{P} = \bar{D} - \epsilon_0 \bar{E} = 0 \qquad \text{For} \qquad d < r < b$$

Check $\quad a < r < d$

$$\bar{P} = \bar{D} - \epsilon_0 \bar{E} = \frac{\hat{a}_r \lambda}{2\pi}\left[\frac{1}{r} - \frac{\epsilon_0}{\epsilon_1 r}\right]$$

$$= \frac{\hat{a}_r \lambda}{2\pi\epsilon_1 r}\left(\epsilon_1 - \epsilon_0\right)$$

$$\rho_{ps}(a^+) = \bar{P}(a^+)\cdot\hat{a}_n = \bar{P}(a)\cdot(-\hat{a}_r) = -\frac{\lambda}{2\pi\epsilon_1 a}\left(\epsilon_1 - \epsilon_0\right)$$

$$\rho_{ps}(d^-) = \bar{P}(d^-)\cdot\hat{a}_n = \bar{P}(d)\cdot(+\hat{a}_r) = \frac{\lambda}{2\pi\epsilon_1 d}\left(\epsilon_1 - \epsilon_0\right)$$

Assuming $\epsilon_1 > \epsilon_0$ we see that the equivalent induced charge on the surface of the inner dielectric "reduces" the charge on the inner metal cylinder and acts similarly for the outer metal cylinder, but with a charge on the dielectric surface at $r = d$.

c) many ways possible, lets try one a bit different to what will be in problem 2.

$$W_e' = \frac{1}{2} C' V_0^2 \,, \qquad ' = \text{properties per Length}$$

If we want to keep the voltage constant to measure the force then

$$\left(\frac{\partial W_e'}{\partial b}\right)_{V_0} = \frac{1}{2} V_0^2 \frac{\partial}{\partial b} C'(b)$$

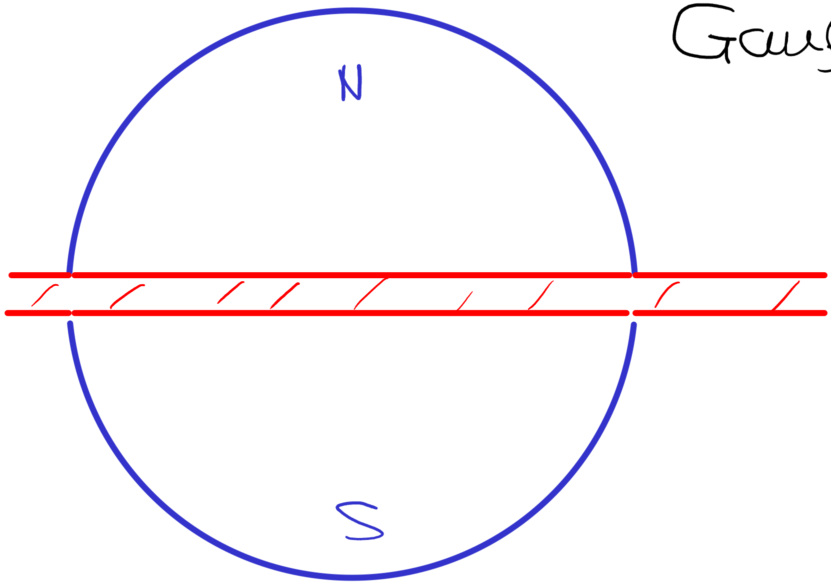
$$= \frac{1}{2} V_0^2 \frac{\partial}{\partial b} \left\{ \frac{2\pi\epsilon_0}{\frac{\epsilon_0}{\epsilon_1} \ln\left(\frac{d}{a}\right) + \ln\left(\frac{b}{d}\right)} \right]$$

$$= -\frac{1}{2} V_0^2 \frac{2\pi\epsilon_0 \frac{1}{b}}{\left\{ \frac{\epsilon_0}{\epsilon_1} \ln\left(\frac{d}{a}\right) + \ln\left(\frac{b}{d}\right) \right]^2}$$

$$= -\frac{1}{2} \frac{\lambda^2}{4\pi^2\epsilon_0^2} 2\pi\epsilon_0 \frac{1}{b} = \boxed{-\frac{\lambda^2}{4\pi\epsilon_0 b}} \quad < 0$$

C decreases as b increases. V_0 kept constant (less charge needed). Total energy per length decreases. Seen like a repulsive force! The boundary condition matters!

Problem 2



Uniform charge distribution ρ total charge Q

Gauss theorem gives

$$\bar{E}(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \dfrac{Q\, \hat{a}_r}{r^2} & r > R \\ \dfrac{1}{4\pi\epsilon_0} Q \dfrac{r\, \hat{a}_r}{R^3} & r \leq R \end{cases}$$

when we pull the hemispheres apart just a tiny bit we add the field in the gap to the set up, consider everything else unchanged. Then we are adding a cylinder to the sphere with a field E

$$W_e = \frac{\epsilon_0}{2} \int d\bar{r}\, E^2$$

but the change in the electrostatic energy will be

$$dW_e = \frac{\epsilon_0}{2} \int E^2 \, 2\pi r \, dr \, dz$$

$$= 2\pi \frac{\epsilon_0}{2} \frac{Q^2}{(4\pi\epsilon_0)^2} \left[\int_0^R \frac{r^3}{R^6} dr + \int_R^\infty \frac{dr}{r^3} \right] dz$$

as we need the change in the field both inside and outside the sphere

$$= \frac{2\pi Q^2 dz}{32\pi^2 \epsilon_0} \left[\frac{R^4}{4R^6} + \frac{1}{2R^2} \right] = \frac{Q^2 dz}{32\pi\epsilon_0} \left[\frac{1}{2R^2} + \frac{1}{R^2} \right]$$

$$= \boxed{+ \frac{3Q^2}{64\pi\epsilon_0 R^2}} \rightarrow$$

The system is an isolated charge distribution (closed system) with no energy from the outside. The mechanical work done by the system dw would be $dw = \overline{F}_Q \cdot d\overline{z}$. This work is done at the expense of the electrostatic energy of the system --> $dw_e = -\overline{F}_Q \cdot d\overline{z}$ --> repulsive force

If a conductor is held at a constant voltage we are dealing with an open system. Charges can then flow in and out of it during a variation of distances and we have to change the formulation. See Andrew Zwangwill pages 144 - 148, and David K. Cheng pages 123 - 126.